Klein - Gordan Equation in Electromagnetic field

An E.M. field can be represent by a vectar potential A and a scalar potential ϕ. These potential form a four vectar A_μ whose component are A_1, A_2, A_3 and $A_4 = i\phi$ and transform like momentum energy four vectar P_μ having component P_1, P_2, P_3 and $P_4 = \frac{iE}{c}$, therefore the potentials A and ϕ should be included in K.G. equation with momentum and Energy

If e is the charge of the particle expresson P and E are replaced by $P - \frac{eA}{c}$ and $E - e\phi$

$$P \rightarrow P - \frac{eA}{c}$$

$$E \rightarrow E - e\phi$$

so the relativistic expression

$$(E - e\phi)^2 - c^2\left(P - \frac{eA}{c}\right)^2 + m^2c^4$$

$$(E - e\phi)^2 = (cP - eA)^2 + m^2c^4$$

replacing E and P by $i\hbar\frac{\partial}{\partial t}$ and $i\hbar\nabla$

$$\left[i\hbar\frac{\partial}{\partial t} - e\phi\right]^2\psi = \left[(-i\hbar c\nabla - eA)^2 + m^2c^4\right]\psi \quad -(1)$$

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)^2\psi = \left[-\hbar^2\frac{\partial^2}{\partial t^2} - ie\hbar\frac{\partial\phi}{\partial t} - ie\hbar\phi\frac{\partial}{\partial t} + e^2\phi^2\right]\psi$$

$$= -\hbar^2\frac{\partial^2\psi}{\partial t^2} - ie\hbar\psi\frac{\partial\phi}{\partial t} - 2ie\hbar\phi\frac{\partial\psi}{\partial t} + e^2\phi^2\psi$$

$$\left(-i\hbar c\nabla - eA\right)^2\psi = i^2\hbar^2c^2\nabla^2\psi + iec\hbar\,\nabla\cdot(A\psi) + iec\hbar A\cdot\nabla\psi + e^2A^2\psi$$

$$= \left[-\hbar^2c^2\nabla^2 + ie\hbar c\nabla\cdot A + 2ie\hbar c A\cdot\nabla + e^2A^2\right]\psi$$

Substituting these values in equ^n - (1)

$$\left[-\hbar^2\frac{\partial^2}{\partial t^2} - ie\hbar\frac{\partial\phi}{\partial t} - 2ie\hbar\phi\frac{\partial}{\partial t} + e^2\phi^2\right]\psi$$
$$= \left[-\hbar^2c^2\nabla^2 + ie\hbar c\nabla\cdot A + 2ie\hbar c A\cdot\nabla + e^2A^2 + m^2c^4\right]\psi \quad -(2)$$

To find the ~~com~~ connection between equ^n - (2) and similar non-relativistic equation. Let us make the following substitution taking mc^2

$$\psi(r,t) = \psi'(r,t)\,e^{-imc^2t/\hbar}$$

$$\frac{\partial\psi}{\partial t} = \left[\frac{\partial\psi'}{\partial t} - \frac{imc^2}{\hbar^2}\psi'\right]e^{-imc^2t/\hbar}$$

$$\frac{d^2\psi}{dt^2} = \left[\frac{d^2\psi'}{dt^2} - \frac{2imc^2}{\hbar}\frac{\partial\psi'}{dt} - \frac{m^2c^4}{\hbar^2}\psi'\right] e^{-imc^2t/\hbar}$$

Substitution these values in eqn - 2

$$\left[-\hbar^2\frac{d^2\psi'}{dt^2} + 2imc^2\hbar\frac{\partial\psi'}{\partial t} + m^2c^4\psi' - ie\hbar\frac{\partial\phi}{\partial t}\psi' - 2ie\hbar\phi\frac{\partial\psi'}{\partial t} - 2mc^2e\phi\psi' + e^2\phi^2\psi'\right]e^{-imc^2t/\hbar}$$

$$= \left[-\hbar^2c^2\nabla^2 + ie\hbar c\nabla\cdot A + 2ie\hbar c A\cdot\nabla + e^2A^2 + m^2c^4\right]\psi'\cdot e^{-imc^2t/\hbar}$$

Cancelling out common factor and both side dividing by $2mc^2$

$$-\frac{\hbar^2}{2mc^2}\frac{\partial^2\psi'}{\partial t} + i\hbar\frac{\partial\psi'}{\partial t} - \frac{ie\hbar}{2mc^2}\frac{\partial\phi}{\partial t}\psi' - \frac{ie\hbar\phi}{mc^2}\frac{\partial\psi'}{\partial t} - e\phi\psi' + \frac{e^2\phi^2}{2mc^2}\psi'$$

$$= \left[\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{2mc}\nabla\cdot A + \frac{ie\hbar}{mc}A\cdot\nabla + \frac{e^2A^2}{2mc^2}\right]\psi'$$

$mc^2 >>$ non relativistic energy E and $mc^2 >> e\phi$ we may neglect in terms of order $1/mc^2$ as compair to E' and $e\phi$ and rearranging

$$i\hbar\frac{\partial\psi'}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{2mc}\nabla\cdot A + \frac{ie\hbar}{mc}A\cdot\nabla + \frac{e^2A^2}{2mc^2} + e\phi\right]\psi'$$

This is non-relativistic eqn for a free particle of charge e in electromagnetic field. Thus K-G. equn ~~is~~ in electro magnetic field reduce to correct non-relativistic limit with appropriate approximation.

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